Sec 7.6 (part 2)

char. poly $(1-\lambda)^2$

Solving $\underline{X}' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{X}$, we found eigenvalue

$\lambda = 1$ has algebraic multiplicity $2.^*$

(* means we expect/hope that $\lambda=1$ has 2 lin. indep eigenvects)

but $\dim(\mathcal{E}_1) = \dim(\mathrm{Null}(\underline{A} - 1I)) = 1 < 2$,

so $\lambda = 1$ has defect $d = 2-1 = 1$. Eigenvect $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $\lambda = 1$

Made $\underline{X}_1 = e^{\lambda t}\underline{v_1}$, but need some

solution $\underline{X}_2$.

If we try $\underline{X}_2 = te^{\lambda t}\underline{v_1}$, does this work? Need $\underline{X}_2' = \underline{A}\underline{X}_2$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\underline{X}_2' = \boxed{e^{\lambda t}\underline{v_1}} + \boxed{\lambda t e^{\lambda t}\underline{v_1}}$$

$$\underline{A}\underline{X}_2 = \underline{A}(te^{\lambda t}\underline{v_1}) = te^{\lambda t}\underline{A}v_1 = te^{\lambda t}\cdot\lambda\underline{v_1}$$

(v_1 is an e-vect $\Rightarrow \underline{A}\underline{v_1} = \lambda\underline{v_1}$)

this did not work for a solution because of extra term $e^{\lambda t}\underline{v_1}$.

Maybe we can add a correction term?

Try $\underline{X}_2 = e^{\lambda t}(t\underline{v_1} + \underline{v_2})$

Check $\underline{X}_2' = \lambda e^{\lambda t}(t\underline{v_1} + \underline{v_2}) + e^{\lambda t}\cdot(\underline{v_1}) = \cancel{\boxed{\lambda t e^{\lambda t}\underline{v_1}}} + \boxed{e^{\lambda t}\underline{v_1} + \lambda e^{\lambda t}\underline{v_2}}$

$\overset{?}{=}$ $\underline{A}\underline{X}_2 = \underline{A}(e^{\lambda t}(t\underline{v_1} + \underline{v_2})) = te^{\lambda t}\underline{A}\underline{v_1} + e^{\lambda t}\underline{A}\underline{v_2}$

$= \cancel{\boxed{\lambda t e^{\lambda t}\underline{v_1}}} + \boxed{e^{\lambda t}\underline{A}\underline{v_2}}$

we get a match $\underline{X}_2' = \underline{A}\underline{X}_2$ iff $e^{\lambda t}\underline{A}\underline{v_2} = \lambda e^{\lambda t}\underline{v_2} + e^{\lambda t}\underline{v_1}$

$$\underbrace{e^{\lambda t} A \underline{v_2}} = \underbrace{\lambda e^{\lambda t} \underline{v_2}} + e^{\lambda t} \underline{v_1}$$

$$\cancel{e^{\lambda t}} \cdot (\underline{A} - \lambda I) \underline{v_2} = \cancel{e^{\lambda t}} \underline{v_1}$$

so we get a match ~~of~~ $\boxed{(A - \lambda I)\underline{v_2} = \underline{v_1}}$.

Since we have $\underline{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, find $\underline{v_2} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$$\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{array}\right] \leftarrow b = 1$$

a free.

choices for $\underline{v_2}$ are $\begin{bmatrix} a \\ 1 \end{bmatrix}$, so pick $\underline{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now we can make solution $\underline{X}_2 = e^{\lambda t}(t\underline{v_1} + \underline{v_2})$

$$= e^{t}\left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= e^{t} \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

$$\cancel{\tfrac{d}{}}\, \underline{X}_1 + \cancel{\tfrac{1}{2}}\underline{X}_2 = e^{\lambda t}\underline{v_1} + \cancel{d}\, e^{\lambda t}(t\underline{v_1} + v_2)$$

$$(1+t) e^{\lambda t} \underline{v_1} + e^{\lambda t} \underline{v_2}$$

have $\underline{X}_1, \underline{X}_2$, so general solution for $\underline{X}' = \underline{A}\underline{X}$ is

$$\underline{X} = c_1 \underline{X}_1 + c_2 \underline{X}_2 = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

- $(A - \lambda I)\underline{v_1} = \emptyset \longrightarrow \underline{v_1}$ is a "true" eigenvector
- $(A - \lambda I)\underline{v_2} = \underline{v_1} \neq \emptyset \longrightarrow \underline{v_2}$ not a "true eigenvector".

OTOH, $\boxed{(A - \lambda I)^2 \underline{v_2}} = (A - \lambda I)\underline{v_1} = \boxed{0}$, so $\underline{v_2}$ is a

"generalized (or virtual) eigenvector"

★ Generalized eigenvectors can also be solutions for

$$(A - \lambda I)^3 \underline{v} = 0, \quad (A - \lambda I)^4 \underline{v} = 0, \ \ldots$$

- For system* $\underline{X}' = A\underline{X}$, if $\underline{A}$ has a "defective eigenvalue" ((alg. mult. λ) > (geom. mult. λ)), then we need to compensate with virtual eigenvectors (as many as the defect)   * 2×1 vector.

- Supposing we have eval λ e-vector (true) $\underline{v_1}$, we need some $\underline{v_2}$ such that
$$\begin{cases} (A-\lambda I)\,\underline{v_1} = 0 \\ (A-\lambda I)\,\underline{v_2} = \underline{v_1} \end{cases}$$

- Once we have these, we make $\underline{X}_1 = e^{\lambda t}\underline{v_1}$
$$\underline{X}_2 = e^{\lambda t}(t\underline{v_1} + \underline{v_2})$$

- General solution is $c_1\underline{X}_1 + c_2\underline{X}_2$.

- <u>How to find virtual eigenvectors</u> (simple case):

(Method 1): Starting w/ $\underline{v_1}$ true eigenvector, find/solve $(A-\lambda I)\,\underline{v_2} = \underline{v_1}$.

(Method 2): (I prefer)

- If we need to make chain $\begin{cases} (A-\lambda I)\,v_1 = 0 \\ (A-\lambda I)\,\underline{v_2} = \underline{v_1} \end{cases}$,
- Find $\underline{v_2}$ such that $(A-\lambda I)^2\underline{v_2} = 0$.
  - If $(A-\lambda I)\underline{v_2} \neq 0$, then "reset" $\underline{v_1} = (A-\lambda I)\underline{v_2}$
  - If $(A-\lambda I)\,\underline{v_2} = 0$, then try a different $\underline{v_2}$.

- If need 2 virtual eigenvectors, then we need a chain
$$\begin{cases} (A-\lambda I)\,\underline{v_1} = 0 \\ (A-\lambda I)\,\underline{v_2} = \underline{v_1} \\ (A-\lambda I)\,\underline{v_3} = \underline{v_2} \end{cases} \longrightarrow \text{"start w/ } (A-\lambda I)^3\underline{v_3} = 0$$